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# UNIDIRECTIONAL FLOWS OF A NON-LINEARLY VISCOUS FLUID IN TUBES†

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A duality is established between the steady unidirectional flow of an incompressible non-linearly viscous fluid and a certain flow of a compressible fictitious gas in a plane perpendicular to the flow velocity vector of the fluid. The functional relationship between the density of the fictitious gas and the velocity is then identical with the relationship between the viscosity and the shear rate. An exact analytic solution of Poiseuille's problem is obtained for a non-linearly viscous Zwanzig–Khon'kin fluid and it is found that no solution exists for Reynolds numbers which exceed a certain critical value. On the basis of this established duality the non-existence of a solution is interpreted as a gas-dynamic "tunnel cutoff" effect, since the Mach number of the fictitious gas then becomes equal to unity on the tunnel walls. 

Consider the steady unidirectional flow with velocity $\mathbf{V} = (0, 0, w(x, y))$ of a Newtonian Reiner–Rivlin fluid [1, 2], induced by a pressure gradient or moving boundaries. The equation of continuity for such a flow is satisfied directly, while the equation of motion takes the form

$$\operatorname{div}\sigma = \nabla p, \quad \sigma = 2\eta D + \mu D^2, \quad 2D = \nabla V + (\nabla V)^T \tag{1}$$

where $p$ is the pressure, $\eta$ and $\mu$ are scalar functions of the invariants of the strain rate tensor $D$, and $\nabla V$ is the tensor of the velocity gradient with components $(\nabla V)_{ij} = \partial V_i/\partial x_j$. It can be shown that for flows of this kind the first and third invariants of the tensor $D$ are equal to zero, while the second $\operatorname{Tr}(D^2) = q^2/2$, $q^2 = (\nabla w)^2$. We then finally obtain

$$\sigma = 2\eta(q)D + \mu(q)D^2 \tag{2}$$

For unidirectional flow $\mathbf{V} = (0, 0, w(x, y))$ we have

$$D = \frac{1}{2}\begin{Vmatrix} 0 & 0 & w_x \\ 0 & 0 & w_y \\ w_x & w_y & 0 \end{Vmatrix}$$

Here $x, y$ and $z$ are Cartesian coordinates, and the $z$ axis is directed along the flow. Assuming that the tensor $\sigma$ also has a similar form, we can directly obtain from (2) that $\mu(q) = 0$. When $\mu(q) \neq 0$ we obtain an overdetermined system of equations in the function $w(x, y)$, which indicates that, in general, the initial assumption that the flow is unidirectional for a Reiner–Rivlin fluid is not satisfied. This agrees with the well-known phenomenon of the onset of a secondary flow in tubes of non-circular cross-section [1, 2]. We will further investigate a Reiner–Rivlin fluid when $\mu = 0$, which is sometimes called a fluid with non-linear viscosity or a generalized Newtonian fluid. Substituting the assumed form for $D$ and $\sigma$ into Eq. (1) we have

$$\operatorname{div}(\eta\mathbf{q}) = 0, \quad \mathbf{q} = \nabla w \tag{3}$$

For simplicity we will consider a flow with a zero pressure gradient. For gradient flows, the first part of Eq. (3) is not equal to zero, which, however, does not change the fundamental qualitative properties of the equation, for example, its type.

We reduce (3) to the form

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$$div(\eta \mathbf{q}) = \eta\lambda^{-1} / [(\lambda - w_x^2)w_{xx} - 2w_x w_y w_{xy} + (\lambda - w_y^2)w_{yy}] = 0$$
$$\lambda = -\eta q dq / d\eta \tag{4}$$

The type of second-order quasilinear differential equation (4) is determined by the sign of the quantity

$$\delta = (\lambda - w_x^2)(\lambda - w_y^2) - w_x^2 w_y^2 = \lambda(\lambda - q^2) \tag{5}$$

whence we conclude that when $\lambda < 0$ the equation is of the elliptic type. This case ($d\eta/dq > 0$) corresponds to dilatant fluids [1] (these include, for example, dense suspensions). When $\lambda > 0$, which corresponds to pseudo-plastic fluids, the situation is more complex. To analyse this case ($d\eta/dq < 0$) further a gas-dynamic analogy will be useful. Note that Eq. (4) is identical (apart from an unimportant factor) with the equation for the velocity potential of the flow of a certain fictitious gas in the $x, y$ plane. The velocity potential $q$ of the gas flow is the component $w(x, y)$ of the flow velocity of the liquid, while the role of the gas density $\rho$ is played by the dynamic viscosity $\eta$. Then, as follows from (4), the velocity of sound $C$ in the fictitious gas is given by the formula

$$C = \lambda^{1/2} \tag{6}$$

We see from (5) that when $\lambda > 0$ a region of ellipticity occurs when $q < C$, while a region of hyperbolicity occurs when $q > C$. As in ordinary gas dynamics, we will introduce a Mach number $M$ given by $M = q/C$. The hyperbolicity regions will then correspond to the condition $M > 1$, while the ellipticity regions will correspond to the condition $M < 1$.

We will consider some examples of fluids with non-linear viscosity. The following power law is widely used [1]

$$\eta = Kq^{n-1} \tag{7}$$

where $n$ and $K$ are constants. Fluids with pseudo-plastic behaviour correspond to $n < 1$. From (4), (6) and (7) we obtain $M^2 = 1 - n$, so that the equations are elliptic for all $q$.

Another form of the functional relationship $\eta(q)$ which has become widely used corresponds to the Prandtl–Eyring model [1], based, at least partially, on molecular representations. It is assumed that

$$\eta = \eta_0 f(qk), \quad f(qk) = \mathrm{arsch}(qk)/(qk)$$

where $\eta$ and $k$ are constants. Simple calculations again show that Eq. (4) is of the elliptic type, i.e. the flow of the fictitious gas is subsonic. The Mach number is then

$$M = \left[1 - ((1 + q^2k^2)^{1/2} f(qk))^{-1}\right]^{1/2}$$

The relationship $M = M(qk)$ is shown in Fig. 1 by the continuous curve, corresponding to subsonic flow.

We will consider another well-known model of a fluid with non-linear viscosity, also obtained taking the kinetic theory into account, namely, the Zwanzig–Khon'kin model [3, 4], for which

$$\eta = \eta_0 \frac{2}{q^2k^2} \mathrm{sh}^2 \xi, \quad \xi = \frac{1}{6} \mathrm{arch}(1 + 9q^2k^2)$$

In this case

$$M = \left[2 - \sqrt{\frac{4q^2k^2}{2 + 9q^2k^2}} \mathrm{cth}\, \xi\right]^{1/2} \tag{8}$$

The corresponding relationship $M = M(qk)$ is shown in Fig. 1 by the dashed curve. It can be seen that Eq. (6) is elliptic when $0 < qk < \sqrt{6}$ and hyperbolic when $qk > \sqrt{6}$. The critical value $q_* k = \sqrt{6}$ for which parabolic degeneracy begins, namely, $M = 1$, can be found from (8) after some calculations. Note that the Mach number of the fictitious gas is bounded: $M < 2\sqrt{3} = 1.1547$. A Tricomi gas has a similar property: $M < 1.14$, while for a perfect gas with adiabatic index $\gamma < 1$, we also have the bound ($M < \sqrt{(2/(1 - \gamma))}$).

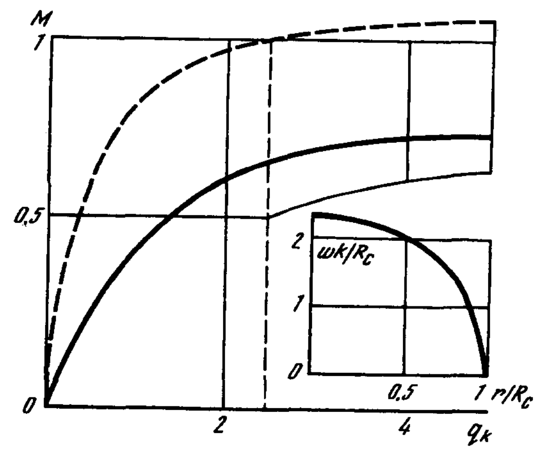


Fig. 1.

Consider the steady flow of a Zwanzig–Khon'kin fluid in a rectilinear tube of arbitrary cross-section produced by a constant pressure gradient (Poiseuille flow). Instead of (3) we then obtain the equation

$$\operatorname{div}(\eta \mathbf{q}) = dp/dz \tag{9}$$

which describes the two-dimensional potential flow of a fictitious gas with uniformly distributed sources. The constant pressure gradient plays the role of the source intensity. We integrate Eq. (9) over the area $S$ of the tube cross-section and change to an integral over the contour $L$ of the cross-section $S$. As a result we obtain

$$\oint \eta \mathbf{q} \cdot \mathbf{n} dl = S \frac{dp}{dz} \tag{10}$$

where $n$ is the unit vector of the normal to the surface of the tube.

To obtain the upper limit of the integral we will attempt to find the maximum of the product $\eta q$. We will write the well-known gas-dynamic relation [5] taking the established duality into account in the form

$$\eta^{-1} d(\rho q)/dq = 1 - M^2$$

Hence we conclude that the product $\eta q$ has an extremum at the critical point $M = 1$, for which, as we already know, $qk = \sqrt{6}$. Using this we obtain the inequality

$$2/(qk)^{-1} \operatorname{sh}^2 \xi \leqslant \sqrt{3/8} \tag{11}$$

the correctness of which can also be verified by a direct check. Using (11), we can rewrite (10) in the form of an inequality, which is the necessary condition for a solution to exist

$$0 \leqslant \mathrm{Re} \leqslant \sqrt{\frac{3}{8}}, \quad \mathrm{Re} = \frac{S}{L} k \left( -\frac{1}{\eta_0} \frac{dp}{dz} \right) \tag{12}$$

It follows from condition (12), in particular, that the Poiseuille flow of a Zwanzig–Khon'kin fluid in a tube with a cross-section in the form of an infinite wedge does not exist. We will isolate in the wedge a sector of radius $R$: $0 \leqslant r \leqslant R$, $|\theta| \leqslant \alpha$. Then the perimeter $L = 2(1 + \alpha)R$, while the area $S = \alpha R^2$. Substituting these quantities into (12) we obtain, for sufficiently large $R$, a contradiction, which also completes the discussion.

For a circular tube the necessary condition (12) is also sufficient. We can verify this by obtaining an exact solution of the problem. Writing Eq. (9) in cylindrical coordinates $(r, \vartheta, z)$ we obtain after integration

$$\operatorname{arch}(1+9q^2k^2) = 6\operatorname{arsh}\left(-\frac{r}{4}\frac{k}{\eta_0}\frac{dp}{dz}\right)^{1/2} \tag{13}$$

Taking the expression ch6x into account in terms of shx, Eq. (16) can be reduced to a quadratic equation, by solving which we finally obtain

$$q = \frac{\sqrt{6}}{k}\frac{R_c}{r}\left\{2\left(\frac{R_c}{r}\right)^2[1-F(r)]-1\right\};\quad F(r) = \left[1-\left(\frac{r}{R_c}\right)^2\right]^{1/2},\quad R_c = \sqrt{\frac{3}{2}}\left(-\frac{k}{\eta_0}\frac{dp}{dz}\right)^{-1} \tag{14}$$

Hence, a solution in fact exists only over a limited range of variation of the tube radius $R$: $0 \leqslant R \leqslant R_c$, in complete agreement with (12). It is interesting to note that the Poiseuille flow of a generalized Newtonian Zwanzig–Khon'kin fluid in a tube of critical cross-section $R = R_c$ corresponds to the fact that the velocity of a fictitious gas on the walls in this case becomes sonic, $M = 1$. In fact, when $r = R_c$, as follows from (14), $qk = \sqrt{6}$, and this in turn corresponds to $M = 1$ (see formula (8) or Fig. 1). The non-existence of a solution when $R > R_c$ resembles the effect, well-known in gas dynamics, of tunnel cutoff, i.e. the fact that it is impossible for the gas to pass the critical cross-section in which a sonic velocity occurs.

Integrating (14), taking the no-slip boundary condition into account, we obtain the fluid flow velocity in the tube

$$w(r) = \frac{\sqrt{6}R_c}{k}\left\{\operatorname{Ln}\frac{r}{R}+\frac{1}{2}\operatorname{Ln}\frac{[1+F(r)][1-F(R)]}{[1-F(r)][1+F(R)]}-\frac{1}{2}\left(\frac{R_c}{r}\right)^2[1-F(r)]+\frac{1}{2}\left(\frac{R_c}{R}\right)[1-F(R)]\right\}$$

In the lower-right part of the figure we show a graph of $w = w(r)$ for the critical tube radius $R/R_c = 1$. The flow rate is equal to

$$Q = \frac{2\sqrt{6}\pi R_c^3}{k}\left[\operatorname{Ln}\frac{R}{R_c}+\frac{1}{32}\left(\frac{R}{R_c}\right)^4 - F(R) - \frac{1}{2}\operatorname{Ln}\frac{1-F(R)}{1+F(R)}+1-\operatorname{Ln}2\right]$$

In the limit as $k \to 0$ or $R_c \to \infty$, which corresponds to a classical Newtonian fluid, the formula obtained corresponds to Poiseuille's law.

The fact that a solution exists was pointed out previously in [6–9] for different flows of a viscoelastic liquid. It was shown in [8, 9], in particular, that Poiseuille flow for the four-constant Oldroyd model [1] only exists over a limited range of Deborah numbers, De

$$0 \leqslant \mathrm{De} \leqslant \frac{1}{\sqrt{c}},\quad \mathrm{De} = \frac{2L}{L}\left(-\frac{\tau}{\eta}\frac{dp}{dz}\right)$$

where $\tau$ is the relaxation time and $c$ is a dimensionless constant of the model.

The breakdown of the solution for a critical value of the dimensionless parameter of the problem (the Reynolds or the Deborah numbers) possibly indicates that under supercritical conditions there are other types of flow, but it more probably confirms that there are certain intrinsic defects of these rheological models. Note also that replacing the usual no-slip boundary condition $w(R) = 0$ by a partial slip condition on the wall $\eta[q(R)]q(R) = -\beta(w_s^2)w_s$, where $w_s$ is the slip velocity, while $\beta(w_s^2)$ is the fluid slip coefficient [2], does not alter the situation in principle: a solution again only exists over a limited range of variation of the tube radius $R$: $0 \leqslant R \leqslant R_c$.

To investigate problems of stability we will consider the simplest unsteady unidirectional flow $\mathbf{V} = (0, 0, w(x, y))$. The equations of motion in this case take the form

$$\frac{\partial w}{\partial t} = \frac{\partial \sigma(q)}{\partial x},\quad q = \frac{\partial w}{\partial x},\quad \sigma = q\eta(q) \tag{15}$$

Differentiating the first equation of (15) with respect to $x$ and using relations (4) and (6) to introduce the Mach number $M$ of the fictitious gas, we obtain the heat-conduction equation

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial x}\left[\eta(1 - M^2)\frac{\partial q}{\partial x}\right] \tag{16}$$

with an effective thermal conductivity $\theta = \eta(1 - M^2)$, which, as we know, is Hadamard unstable when $\theta < 0$ (see, for example, [10]), i.e. when $M > 1$.

The non-existence of a solution of the steady problem considered when $M > 1$ manifests itself in the fact that the corresponding unsteady problem is non-evolutionary in the hyperbolicity region, although, in general, there is no basis for asserting that a change of type is impossible in the region of evolution. The fact that the equations are non-evolutionary indicates that, in this region, the model employed is no longer correct. It is possible that to eliminate the defect in the rheological model one needs to take into account additional physical factors, which may be small in the region of evolution but play a decisive role in the region of non-evolution. The fact that the equations of motion of a fluid with a non-linear viscosity are non-evolutionary in the hyperbolicity region was pointed out in [11].

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